Kinks and solitons in SUSY models

1990 J. Phys. A: Math. Gen. 23 1645

(http://iopscience.iop.org/0305-4470/23/9/025)

J. Phys. A: Math. Gen. 23 (1990) 1645-1656. Printed in the UK

Kinks and solitons in SUSY models

L J Boya and J Casahorran
Departamento de Fisica Teorica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

Received 29 August 1989

Abstract. We consider arbitrary two-dimensional supersymmetric theories including kinks or solitons solutions. Going to sine-Gordon and $(\lambda\phi^4)_{1+1}$ theories, we compute the first quantum correction to the classical mass using a technique which only needs the discrete levels of Schrödinger equations. Attention is devoted to scattering and statistics properties of soliton solutions of the SUSY sine-Gordon equation.

1. Introduction

Non-perturbative techniques have been used widely in quantum field theories over the last few years. In particular, solitary-wave solutions of nonlinear field equations indicate the existence of topological quantum sectors in the global theory [1]. In fact, such classical solutions appear to be related to a new particle of the model, normally called the 'baryon', which in the weak-coupling regime always exhibits a large mass in comparison with the mesons constructed out of the homogeneous vacuum. Restricting ourselves to bidimensional models, we can consider the so-called kink solutions as well as the more restrictive family of solitons.

Simultaneously enormous progress has been achieved in a disconnected realm of QFT. We allude to supersymmetric theories with a new kind of symmetry which unifies bosonic and fermionic quanta [2]. In order to analyse the spontaneous breakdown of SUSY, we can consider the simplest possible model, namely that of a chiral field associated with the $N=1$ general case, in $1+1$ dimensions [3]. Using the powerful superfield formalism, we formulate a general SUSY invariant action where a clever choice for the so-called superpotential function allows us to consider systems including kinks or solitons as classical solutions. In particular the sine-Gordon and $(\lambda\phi^4)_{1+1}$ models are easily recovered once the adequate choice of superpotential function is made.

Now we recall one of the most exciting properties of supersymmetric field theories, namely that the invariance of the vacuum under SUSY transformations appears connected to the vanishing of the vacuum energies. It is known that the vacuum diagrams are cancelled among bosonic and fermionic loops to all orders around the classical vacuum [4]. Working over the background provided by the inhomogeneous classical solutions of bidimensional models, the non-vanishing of the quantum correction to the classical mass is traced to the existence of supersymmetric violating boundary terms in the Lagrangian. In fact, the SUSY content of the model reduces to the $N=\frac{1}{2}$ case instead of the initial $N=1$ situation [5].

Many authors have computed the one-loop correction to the classical mass of kinks or solitons under the restriction to a finite box of length L by obtaining the eigenvalue spectrum and then, once the sum of these eigenvalues is performed, allowing the dimension of the box to tend to infinity [6]. However, simple expressions can be found for bosonic corrections in some particular cases (we will discuss the two required conditions). The above-mentioned method incorporates the renormalisation counter-terms, a normal ordering prescription being scalar bidimensional models, and curiously only uses the discrete levels of the stability equation (a conventional Schrödinger operator [7]). In this paper we extend this bosonic technique to the first fermionic correction to the classical mass. The point is the relation between bosonic and fermionic density of states per wavenumber. The unified treatment is then applied to the kink solution of $(\lambda\phi^4)_{1+1}$ theory and the soliton solution of the sine-Gordon model.

In particular the sine-Gordon system exhibits interesting properties. One of the most notorious is the equivalence to the massive Thirring model [8], a surprising fact because the latter is expressed in terms of fermionic fields only. Even more surprising, the sine-Gordon model is equivalent to a free massive Dirac theory whenever a certain combination of coupling constants fulfils a special condition. Pursuing this analogy, the soliton represents a particle with fermion number 1 while the antisoliton adopts the value -1. In fact, the soliton and the antisoliton appear as the lightest particles in the theory with non-zero fermion number. Therefore the SG soliton is nothing other than the fundamental fermion of the massive Thirring model. Anyway, we see a fermion appearing as a coherent state of a Bose field without any conflict with the hypothetical spin-statistics theorem, since in one dimension there is no spin. Using a semiclassical description, based on a generalisation of the well known relation between phase shift and time delay in potential theory, the different collision channels for the sine-Gordon theory have been studied [9]. Restricting ourselves to soliton–soliton scattering, we have a repulsive force with part of the interaction region excluded for the particles, with the result that we find a time advance due to the shorter distance each particle travels. Moreover, the absence of bound states reinforces the preceding interpretation and gives us a nice place to apply the exclusion principle, the SG soliton being a fermion.

Going to a SUSY version of the SG model, we can extend the conventional analysis. We recall the bosonic stability equation where an unavoidable zero-energy mode emerges due to translational invariance. Invoking SUSY, this zero mode percolates into the fermionic part of our model. In a conventional interpretation we should consider a state with the zero mode occupied and a second one where that state remains unoccupied. Then it is shown that the fermion number is generally transferred between the soliton and the (anti) soliton in the scattering process. Taking the soliton–soliton channel in which the zero-energy fermion state is occupied for both, study of the S-matrix in the semiclassical approximation reveals the emergence of a bound state. This fact supports the interpretation of the SUSY sine-Gordon soliton as a boson on the understanding that the fermionic zero mode is occupied. When this state remains unoccupied the fermionic character of the soliton is recovered.

The paper is arranged in the following way. Section 2 is devoted to a general description of SUSY models including kinks or solitons, while the third section includes the first quantum corrections to the classical mass of the topological object. Then, in section 4, we analyse the scattering process, including arguments about the statistics of solitons in the SUSY SG model. Our main conclusions are finally stated in section 5.

2. SUSY models including kinks or solitons

In order to establish the general scheme we consider a model constructed out of the single chiral scalar superfield in $1+1$ dimensions. The interaction terms can take a much more general form than in the conventional $2+1$ or $3+1$ cases, due to the renormalisability of the different models. In the following we will use the powerful superfield formalism [10]. As we are interested in a SUSY $N=1$ theory, the real spinorial anticommuting parameters correspond to

$$\xi^{\alpha}=\varepsilon^{\alpha\beta}\xi_{\beta} \tag{2.1}$$

$$\xi^{\alpha\beta}=-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{2.2}$$

$$(\xi\eta)=\xi^{\alpha}\eta_{\alpha}=\varepsilon^{\alpha\beta}\xi_{\beta}\eta_{\alpha}. \tag{2.3}$$

Now we consider the chiral scalar superfield, namely

$$\Phi(x,\theta)=\phi(x)+\mathrm{i}\theta\Psi(x)+\tfrac{1}{2}\mathrm{i}(\theta\theta)F(x) \tag{2.4}$$

where ϕ is a real scalar field, Ψ is a Majorana field and F represents the so-called 'auxiliary field'. Passing to the superspace $(x,\theta)=(x^0,x^1,\theta^0,\theta^1)$, we must introduce the covariant derivatives

$$Q_{\alpha}=\frac{\partial}{\partial\theta^{\alpha}}+(\gamma^{\mu}\theta)_{\alpha}\partial_{\mu}\qquad \bar{Q}_{\alpha}=\mathrm{i}\varepsilon^{\alpha\beta}Q_{\beta} \tag{2.5}$$

together with their conjugate derivatives

$$D_{\alpha}=\frac{\partial}{\partial\theta^{\alpha}}-(\gamma^{\mu}\theta)_{\alpha}\partial_{\mu}\qquad \bar{D}_{\alpha}=\mathrm{i}\varepsilon^{\alpha\beta}D_{\beta}. \tag{2.6}$$

Defining

$$P_{\mu}=-\mathrm{i}\partial_{\mu} \tag{2.7}$$

we recover the conventional superalgebra associated with the $N=1$ case, namely

$$\{Q_{\alpha},\bar{Q}_{\beta}\}=2(\gamma^{\mu}P_{\mu})_{\alpha\beta}\qquad [Q_{\alpha},P_{\mu}]=0\qquad [\bar{Q}_{\alpha},P_{\mu}]=0. \tag{2.8}$$

Moreover

$$\{Q_{\alpha},D_{\beta}\}=0. \tag{2.9}$$

As a matter of fact, the actions we are looking for represent particular cases of a SUSY σ model [11]. The action of the free theory is now

$$S_0=\frac{\mathrm{i}}{4}\int[\Phi(x,\theta)(D^{\alpha}D_{\alpha})\Phi(x,\theta)]\,\mathrm{d}^2x\,\mathrm{d}^2\theta \tag{2.10}$$

which easily reduces to the well known form

$$S_0=\tfrac{1}{2}\int[(\partial_{\mu}\phi)^2+\bar{\Psi}(\mathrm{i}\gamma^{\mu}\partial_{\mu})\Psi+F^2]\,\mathrm{d}^2x. \tag{2.11}$$

Bearing in mind the special renormalisability properties of scalar bidimensional models, we can choose an interacting part of the action to be

$$S_1=\tfrac{1}{2}\int W[\Phi(x,\theta)]\,\mathrm{d}^2x\,\mathrm{d}^2\theta \tag{2.12}$$

which, once we perform the conventional expansion, leads to

$$S_1 = \tfrac{1}{2}\int (2FW' - W''\bar{\Psi}\Psi)\,\mathrm{d}^2x \tag{2.13}$$

where $W(\phi)$ is the so-called superpotential function and the prime denotes a derivative with respect to the argument. Combining both contributions, the total action is

$$S = \tfrac{1}{2}\int [(\partial_\mu\phi)^2 + \bar{\Psi}(\mathrm{i}\gamma^\mu\partial_\mu)\Psi + F^2 + 2FW' - W''\bar{\Psi}\Psi]\,\mathrm{d}^2x. \tag{2.14}$$

The classical Euler–Lagrange equation for the 'auxiliary field' F is simply

$$F = -W' \tag{2.15}$$

and hence, on elimination of F, (2.14) becomes

$$S = \tfrac{1}{2}\int [(\partial_\mu\phi)^2 + \bar{\Psi}(\mathrm{i}\gamma^\mu\partial_\mu)\Psi - W'^2 - W''\bar{\Psi}\Psi]\,\mathrm{d}^2x. \tag{2.16}$$

In particular, (2.15) is concerned with the spontaneous breakdown of supersymmetry. That phenomenon occurs if and only if $F \neq 0$ at the potential minimum [12]. Once we have chosen a superpotential function $W(\phi)$ such that the associated theory admits topological classical solutions, the SUSY algebra must be supplemented by the central charges [13].

After producing a theory which exhibits topological classical solutions, we can perform the quantisation programme by perturbing around those solutions, always in a 'weak-coupling' approximation. In particular we are interested in the first quantum correction to the classical mass, using methods which normally proceed via the semiclassical approximation or one-loop terms. Restricting to the bosonic part, many authors have computed the one-loop contribution under the restriction to a finite space of length L; once having studied the eigenstate spectrum, one can sum the eigenstates. The last step allows the length of the box to be taken to infinity [6]. In some particular cases, including the $(\lambda\phi^4)_{1+1}$ theory and the sine-Gordon model, the general expression reduces to a simple formula which only includes the discrete levels of Schrödinger equations, once the renormalisation counterterms have been incorporated. The objective pursued in this section is the extension of the above-mentioned technique to the fermionic contribution, thus resulting in a unified treatment particularly useful in SUSY models. Taking a general theory, we start by shifting the scalar field by the classical solution $\phi_c(x)$, $\phi(x) = \phi_c(x) + \varphi(x)$; these bosonic fluctuations obey the stability equation [5]

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tfrac{1}{2}[W'(\phi_c)^2]''\right)\varphi(x) = \omega_B^2\varphi(x) \tag{2.17}$$

whenever we consider the conventional Bogomolny condition

$$\frac{\mathrm{d}\phi_c(x)}{\mathrm{d}x} = \pm W'(\phi_c). \tag{2.18}$$

Writing the spinor in its two-component form

$$\Psi(x) = \begin{bmatrix} u_+(x) \\ u_-(x) \end{bmatrix} \tag{2.19}$$

the Dirac equation becomes

$$Q^{+}u_{-}=\mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x}+W''(\phi_{c})\right)u_{-}(x)=-\omega_{F}u_{+}(x) \tag{2.20a}$$

$$Qu_{+}=\mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x}-W''(\phi_{c})\right)u_{+}(x)=-\omega_{F}u_{-}(x) \tag{2.20b}$$

which, making full use of the hidden SUSY quantum mechanical character of the Dirac equation over the background provided by $\phi_c(x)$, yields

$$Q^{+}Qu_{+}=\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+W''(\phi_{c})^2+\frac{\mathrm{d}W''(\phi_{c})}{\mathrm{d}x}\right)u_{+}(x)=\omega_{F}^2u_{+}(x) \tag{2.21a}$$

$$QQ^{+}u_{-}=\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+W''(\phi_{c})^2-\frac{\mathrm{d}W''(\phi_{c})}{\mathrm{d}x}\right)u_{-}(x)=\omega_{F}^2u_{-}(x) \tag{2.21b}$$

so that one of the fermionic components (depending on the sign of (2.18)) fulfils the bosonic fluctuation equation (2.17). Now the first quantum correction to the classical mass is given by

$$\Delta M=\tfrac{1}{2}\sum(\omega_{B}-\omega_{F}). \tag{2.22}$$

In order to write a more rigorous version of (2.22) we need the bosonic (fermionic) densities of states n_B (n_F) in the continuous part of the spectrum. Let us also define n_+ (n_-) to be the densities of states of the operators Q^+Q (QQ^+). Depending on the sign of (2.18), we have

$$n_B=n_+ \qquad \text{or} \qquad n_B=n_- \tag{2.23}$$

while

$$n_F=\tfrac{1}{2}(n_++n_-). \tag{2.24}$$

This relation between n_F and $n_\pm$ requires a further analysis [5]. As n_F is the density associated with the fermionic operator

$$F=\begin{bmatrix}0 & Q^{+}\\ Q & 0\end{bmatrix} \tag{2.25}$$

it represents half the density of

$$F^2=\begin{bmatrix}Q^{+}Q & 0\\ 0 & QQ^{+}\end{bmatrix}. \tag{2.26}$$

To prove this fact we consider the operator

$$P=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{2.27}$$

which anticommutes with F, while it commutes with F^2. For each eigenvalue ε^2 of F^2 we can find two eigenstates $|\Omega\rangle$ and $P|\Omega\rangle$, only one of which is a positive-frequency eigenstate of F with eigenvalue ε. If we bear in mind the densities of the operators Q^+Q (QQ^+), equation (2.24) is recovered. The final conclusion of the preceding argument reduces to

$$n_B-n_F=\pm\tfrac{1}{2}(n_+-n_-) \tag{2.28}$$

so that the first quantum correction to the classical mass is

$$\Delta M = \pm\tfrac{1}{4}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{2\pi}\left(\frac{\mathrm{d}n_+}{\mathrm{d}t}-\frac{\mathrm{d}n_-}{\mathrm{d}t}\right)(t^2+W''(\phi_0)^2)^{1/2} \tag{2.29}$$

where $W''(\phi_0)$ corresponds to the mass of the homogeneous vacuum ϕ_0. Invoking the supersymmetry properties of the model, the discrete eigenvalue contribution cancels. In fact, the non-zero character of ΔM is due to the different density of states n_+ (n_-) in the continuous part of the spectrum. Restricting ourselves to the cases where the potentials $U(x)$ which appear in (2.21) obey two conditions, namely that the proper $U(x)$ be 'reflectionless' and that $(1+|x|)U(x)$ be integrable, ΔM reduces to a very simple form that only involves the discrete spectrum [7]. In order not to clutter the paper, we omit any discussion of the method and its peculiarities. Anyway, a detailed exposition can be found in [14]. We simply point out the final result concerning (2.29). If we split the ΔM general contribution in the form

$$\Delta M = \pm(\Delta M_+ - \Delta M_-) \tag{2.30}$$

the partial contributions can be calculated using [7]

$$\Delta M_j = -\frac{W''(\phi_0)}{2\pi}\sum_i(\sin\theta_{ji}-\theta_{ji}\cos\theta_{ji}) \qquad j=+,- \tag{2.31}$$

where the sum extends over the discrete levels of (2.21) below the starting point of the continuous spectrum $W''(\phi_0)$, ϕ_0 being the homogeneous vacuum. Moreover, the θ_{ji} are given by

$$\theta_{ji} = \cos^{-1}\left(\frac{\omega_{ji}}{W''(\phi_0)}\right) \qquad 0\leqslant\omega_{ji}\leqslant W''(\phi_0). \tag{2.32}$$

Returning to (2.21), we recognise the typical supersymmetric quantum mechanics exercise where the superpotential corresponds to $W''(\phi_c)$. In particular we recall the duplication of non-zero eigenvalues in (2.21) while the zero-energy mode can appear as a singlet. According to the sign of the Bogomolny condition (2.18), the bosonic equation (2.17) coincides either with (2.21*a*) or with (2.21*b*). If we bear in mind the mandatory zero-energy mode of (2.17) due to translational invariance [6], it happens that our SUSYQM problem exhibits a zero mode. In these conditions (2.30) reduces to the zero-energy contribution, namely

$$\Delta M = -\frac{W''(\phi_0)}{2\pi}. \tag{2.33}$$

3. Two examples: $(\lambda\phi^4)_{1+1}$ and sine-Gordon

Now we concentrate on the most typical models exhibiting kinks or solitons. In particular we consider the SUSY versions of the $(\lambda\phi^4)_{1+1}$ and sine-Gordon theories.

3.1. The $(\lambda\phi^4)_{1+1}$ theory

It suffices to take

$$W'(\phi) = \left(\frac{\lambda}{2}\right)^{1/2}\left(\phi^2-\frac{m^2}{\lambda}\right) \tag{3.1}$$

a choice which leads to the action

$$S=\tfrac{1}{2}\int\left[(\partial_\mu\phi)^2+\bar{\Psi}(\mathrm{i}\gamma^\mu\partial_\mu)\Psi-\frac{\lambda}{2}\left(\phi^2-\frac{m^2}{\lambda}\right)^2-\sqrt{2\lambda}\,\phi\bar{\Psi}\Psi\right]\mathrm{d}^2x. \tag{3.2}$$

Looking for the potential minima, we can label the classical vacua as

$$\langle\phi\rangle_-=-\frac{m}{\sqrt{\lambda}}\qquad \langle\phi\rangle_+=\frac{m}{\sqrt{\lambda}}. \tag{3.3}$$

Returning to (2.15), we recover the unbroken supersymmetry character of the model. Expanding the scalar field around the VEV (vacuum expectation value), $\phi(x)=\pm m/\sqrt{\lambda}+\varphi$, we recover the mass degeneracy

$$m_\varphi^2=m_\Psi^2=2m^2. \tag{3.4}$$

Moreover, the kink solution adopts the form [6]

$$\phi_{\mathrm{k}}(x)=\frac{m}{\sqrt{\lambda}}\tanh\left(\frac{mx}{\sqrt{2}}\right) \tag{3.5}$$

while the solution with the minus sign in front will represent the antikink. If we recall the classical energy density of the kink

$$\varepsilon(x)=\frac{1}{2}\left(\frac{\mathrm{d}\phi_{\mathrm{k}}}{\mathrm{d}x}\right)^2+\tfrac{1}{2}W'^2(\phi_{\mathrm{k}}) \tag{3.6}$$

we can calculate the classical mass associated with the extended object

$$M_{\mathrm{cl}}=\int_{-\infty}^{\infty}\varepsilon(x)\,\mathrm{d}x=\frac{2\sqrt{2}m^3}{3\lambda}. \tag{3.7}$$

Returning to the results obtained in section 2, the Schrödinger equation satisfied by the bosonic quantum fluctuations $\varphi(x)$ is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+3m^2\tanh^2\left(\frac{mx}{\sqrt{2}}\right)-m^2\right)\varphi(x)=\omega_{\mathrm{B}}^2\varphi(x) \tag{3.8}$$

while the couple (2.21) reduces to

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+m^2\tanh^2\left(\frac{mx}{\sqrt{2}}\right)+m^2\right)u_+(x)=\omega_{\mathrm{F}}^2u_+(x) \tag{3.9a}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+3m^2\tanh^2\left(\frac{mx}{\sqrt{2}}\right)-m^2\right)u_-(x)=\omega_{\mathrm{F}}^2u_-(x) \tag{3.9b}$$

so that (3.8) and (3.9*b*) coincide in accordance with the Bogomolny condition for the kink solution of $(\lambda\phi^4)_{1+1}$ theory, namely

$$\frac{\mathrm{d}\phi_{\mathrm{k}}(x)}{\mathrm{d}x}=W'(\phi_{\mathrm{k}}). \tag{3.10}$$

Therefore the zero-mode appears associated with $u_-(x)$ while the first quantum correction to the classical mass is easily computed using (2.33)

$$\Delta M=-\frac{\sqrt{2}m}{2\pi}. \tag{3.11}$$

3.2. The sine-Gordon system

Now we are concerned with

$$W'(\phi)=\frac{2m^2}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}\,\phi}{2m}\right) \tag{3.12}$$

and therefore

$$S=\tfrac{1}{2}\int\left\{(\partial_\mu\phi)^2+\bar{\Psi}(\mathrm{i}\gamma^\mu\partial_\mu)\Psi-\frac{2m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}\,\phi}{m}\right)\right]-m\cos\left(\frac{\sqrt{\lambda}\,\phi}{2m}\right)\bar{\Psi}\Psi\right\}\mathrm{d}^2x. \tag{3.13}$$

According to the preceding treatment, the absolute minima of the potential correspond to

$$\frac{\sqrt{\lambda}\,\phi}{m}=2m\pi \qquad m\in\mathbb{Z} \tag{3.14}$$

a limit case with infinite vacua and $\mathbb{Z}$ field translational symmetry. Although a complete analysis of the theory constructed out of the homogeneous vacua of (3.14) would require considerable work, we recall the pure bosonic case where the doublet state represents the fundamental particle of the sector [6], we can obtain a simple picture by expanding the density Lagrangian in powers of the coupling constant λ. We then get

$$\frac{m^4}{\lambda}\left(1-\cos\frac{\sqrt{\lambda}\,\phi}{m}\right)\simeq\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4+\ldots \tag{3.15a}$$

$$\frac{m}{2}\cos\left(\frac{\sqrt{\lambda}\,\phi}{2m}\right)\bar{\Psi}\Psi\simeq\frac{m}{2}\bar{\Psi}\Psi-\frac{\lambda}{2m}\phi^2\bar{\Psi}\Psi+\ldots \tag{3.15b}$$

thus confirming the mass degeneracy and the unbroken SUSY character of the model around the homogeneous vacua. To proceed to section 4 we need the soliton solution [6], namely

$$\phi_{\mathrm{s}}(x)=\frac{4m}{\sqrt{\lambda}}\tan^{-1}[\exp(mx)] \tag{3.16}$$

as well as the antisoliton

$$\phi_{\bar{\mathrm{s}}}(x)=-\frac{4m}{\sqrt{\lambda}}\tan^{-1}[\exp(mx)]. \tag{3.17}$$

Moreover, the classical energy associated with both topological solutions is simply

$$M_{\mathrm{cl}}=\frac{8m^3}{\lambda}. \tag{3.18}$$

Going now to the bosonic fluctuations $\varphi(x)$, we find

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2m^2\tanh^2 mx-m^2\right)\varphi(x)=\omega_{\mathrm{B}}^2\varphi(x) \tag{3.19}$$

while the fermionic components correspond to

$$\left(-\frac{d^2}{dx^2}+2m^2\tanh^2 mx-m^2\right)u_+(x)=\omega_F^2 u_+(x) \tag{3.20a}$$

$$\left(-\frac{d^2}{dx^2}+m^2\right)u_-(x)=\omega_F^2 u_-(x). \tag{3.20b}$$

The coincidence between (3.19) and (3.20*a*) is easily understood bearing in mind the Bogomolny condition

$$\frac{d\phi_s(x)}{dx}=-W'(\phi_s) \tag{3.21}$$

with the following first quantum correction to the classical mass:

$$\Delta M=-\frac{m}{2\pi}. \tag{3.22}$$

In this way the non-vanishing contribution to the energy can be computed via a unified treatment which only considers the zero-mode terms. If we recall the pure bosonic case, where the quantum correction is calculated by a sum over the discrete spectrum, the fundamental point is that the SUSY character of the models simplifies the formula obtained in [7].

4. Scattering and statistics of solitons in SUSY sine-Gordon theory

As early as the 1960s Skyrme [15] suggested that the sine-Gordon soliton, despite of arising from a bosonic field theory, may be equivalent to fermions which interact through a four-fermion term. More recently Coleman [8] established such an equivalence within the framework of perturbation theory. In fact the sine-Gordon system turns out to be equivalent to the massive Thirring model (if a certain combination of coupling constants fulfils a special condition, SG corresponds with a free massive Dirac theory). Thus the soliton is a particle of fermion number 1 while the antisoliton represents the particle of fermion number -1. Moreover, the soliton and the antisoliton appear as the lightest particles in the theory with non-zero fermion number. Therefore a fermion emerges as a coherent state of a Bose field, a result which could not happen in three space dimensions, for example, due to the spin-statistics theorem. However, we cannot apply the spin-statistics theorem in one dimension since then there is no spin.

In order to discuss statistical properties the scattering process provides valuable information. Following a semiclassical approximation, based on the relation between phase shift and time delay in simple potential theory, the description of soliton scattering is available. On the other hand, when the time delay in a collision process is known, the sine-Gordon model constitutes an excellent example, the phase shifts and hence the S-matrix elements can be determined easily. To sum up, we can write the equation for the semiclassical phase shift [9]

$$\delta(E)=\frac{n_B\pi}{2}+\frac{1}{2}\int_{E_{th}}^{E}\Delta t(E')\,dE' \tag{4.1}$$

where Δt represents the classical time delay in the collision while n_B points out the number of bound states below the threshold energy E_{th}. Restricting the analysis to the pure sine-Gordon model, where the time delay for soliton-antisoliton and soliton-soliton scattering is well known [6], the phase shifts are [9]

$$\delta_{s\bar{s}}(u)=\frac{4\pi^2m^2}{\lambda}+\frac{16m^2}{\lambda}\int_0^u \frac{\ln x}{1-x^2}\,\mathrm{d}x \tag{4.2a}$$

$$\delta_{ss}(u)=\frac{16m^2}{\lambda}\int_0^u \frac{\ln x}{1-x^2}\,\mathrm{d}x \tag{4.2b}$$

on the understanding that $n=8\pi m^2/\lambda$ is the maximum number of soliton-antisoliton bound states. In particular, this soliton-antisoliton channel scattering admits a simple physical interpretation. If we recall the asymptotic form of the classical solution, the $\Delta t<0$ value, namely [9]

$$\Delta t(u)=\frac{2}{mu\gamma}\ln u \qquad \gamma=\frac{1}{\sqrt{1-u^2}} \tag{4.3}$$

(where u is the asymptotic velocity of each particle) indicates attractive forces, a physical picture in accordance with the emergence in this soliton-antisoliton channel of a certain number of bound solutions. On the other hand, the soliton-soliton scattering requires a careful analysis. The asymptotic behaviour of the classical solution admits a double physical interpretation. In principle one might accept a picture very similar to the one pointed out for the soliton-antisoliton channel with total transmission and negative time delay (see (4.3)). This possibility can be rejected at once because it would mean the presence of attractive forces in a channel where the bound states are absent. However, (4.2b) admits a different interpretation in terms of a total reflection phenomenon with the time delay which appears in (4.3). More precisely, if the forces are repulsive and therefore a part of the interaction region is excluded from the particles, the time advance is a logical result due to the shorter distance each particle travels. In this case the absence of bound states needs no additional comment. Anyway this picture of backward soliton-soliton scattering provides a natural place for the semi-classical version of the Pauli exclusion principle, thus confirming the common knowledge that the soliton solution of SG theory represents a fermion.

Now we want to extend the preceding scheme to the supersymmetric version of the model. If know the S-matrix for soliton scattering in the semiclassical approximation, including the contribution provided by the fermion zero mode and its transference process, a more complete analysis of the statistics properties is at hand. Restricting ourselves for the moment to studying the case of the (anti) soliton only, we find a fermion zero-energy mode as the following:

$$\Psi_0(x)=C\begin{bmatrix}(\cosh mx)^{-1}\\ 0\end{bmatrix} \tag{4.4}$$

where C represents a finite normalisation constant. Now the fermion field operator may be expanded as

$$\Psi(x,t)=\sum_k\left[b_k u_k(x)\exp(-\mathrm{i}\omega_k t)+b_k^+u_k^*(x)\exp(\mathrm{i}\omega_k t)\right] \tag{4.5}$$

where $u_k(x)$ is the conventional spinor while the operators b_k (b_k^+) satisfy anticommutation relations

$$\{b_k, b_{k'}^+\}=\delta_{kk'}. \tag{4.6}$$

In particular the fermion number can be computed using the operator

$$N=(b_0^+b_0-\tfrac{1}{2})+\sum_k b_k^+b_k. \tag{4.7}$$

Since the operator b_0 is associated with the fermion zero mode, it happens that operating on the ground state with b_0^+ produces another state with identical energy. As we cannot distinguish occupied and unoccupied states by energy for the fermion zero mode, there are two degenerate states which have fermion number different by a unit. Working on the soliton (antisoliton), the two above-mentioned states will satisfy

$$b_0|-;\, s(\bar{s})\rangle=0 \qquad b_0^+|-;\, s(\bar{s})\rangle=|+;\, s(\bar{s})\rangle \tag{4.8}$$

so that, following the conventional interpretation, the 'plus' state will be called occupied and the 'minus' state unoccupied. We can conclude that over the background provided by the soliton (antisoliton) the fermion presence leads to a doubly degenerate lowest-energy state.

Now we can pass to the transfer of fermion number in the scattering process. Starting from the explicit soliton–antisoliton and soliton–soliton solutions, the general Dirac equation can be analytically solved whenever the constants of the model fulfil a certain condition [16]. (In fact this special condition reduces the general system to our supersymmetric case.) Using the typical 'in' ('out') states, the complete analysis of the scattering processes is available. While a detailed exposition of this subject can be found in [16], we point out simply the results we need to discuss the statistical properties. To sum up, the S-matrix for soliton–antisoliton scattering, including the fermion zero-mode contribution, is given by (see section 3 of [16])

$$S_{s\bar{s}}=(2\pi)^2\delta^{(2)}(p_i-p_f)\exp(2i\delta_{s\bar{s}})\left\{\begin{array}{l}{}_{\rm in}\langle s\bar{s};\,--|--;\,s\bar{s}\rangle_{\rm out}=1\\ {}_{\rm in}\langle s\bar{s};\,++|++;\,s\bar{s}\rangle_{\rm out}=-1\end{array}\right\} \tag{4.9}$$

where p_i (p_f) represent the asymptotic momenta of the particles while the factor $\exp(2i\delta_{s\bar{s}})$ includes the bosonic contribution presented in (4.2*a*). We can deal with soliton–soliton scattering in a similar manner to that outlined for a soliton–antisoliton collision. In fact

$$S_{ss}=(2\pi)^2\delta^{(2)}(p_i-p_f)\exp(2i\delta_{ss})\left\{\begin{array}{l}{}_{\rm in}\langle ss;\,--|--;\,ss\rangle_{\rm out}=1\\ {}_{\rm in}\langle ss;\,++|++;\,ss\rangle_{\rm out}=-1\end{array}\right\}. \tag{4.10}$$

With these data to hand we can discuss the effect of the fermion presence on the scattering of solitons. Whenever the zero mode remains unoccupied, see (4.10), the physical picture already considered maintains its validity, i.e. the soliton exhibits a fermionic character consistent with the scattering results. However, the situation is disturbed when considering solitons with occupied fermion zero modes. Unlike the former case, a smart 'minus' sign emerges in (4.10). Incorporating this contribution into the exponential factor, we find a modified phase shift, namely

$$\delta'_{ss}(u)=\frac{\pi}{2}+\frac{16m^2}{\lambda}\int_0^u\frac{\ln x}{1-x^2}\,dx. \tag{4.11}$$

If we bear in mind the general formula (4.1) the soliton–soliton scattering, with occupied fermion zero modes, includes a time advance (see (4.3)) which indicates attractive forces. Moreover, this fact is consistent with the existence of a bound state in the channel. The former arguments support the interpretation of the SUSY sine-Gordon soliton as a boson on the understanding that the fermionic zero mode is occupied. When this state remains unoccupied the fermionic character of the soliton is recovered.

5. Conclusions

Using the powerful superfield formalism, we have considered arbitrary two-dimensional supersymmetric theories. Simple choices for the superpotential function allow us to recover the most well known cases: $(\lambda\phi^4)_{1+1}$ and sine-Gordon. As a matter of fact, we can compute the first quantum correction to the classical mass by means of a simple technique which only uses the discrete levels of Schrödinger equations. Invoking SUSY, cancellation occurs for non-zero eigenvalues with a final term associated with zero-energy modes. Attention is also devoted to scattering and statistical properties of solitons in SUSY sine-Gordon. In particular, we complete the qualitative arguments which in pure sine-Gordon theory identify the soliton as a fermionic particle.

Acknowledgment

We acknowledge partial support from the Comisión de Investigación Científico y Técnica (Spain).

References

[1] Christ N H and Lee T D 1975 *Phys. Rev.* D **12** 1606
Dashen R F, Hasslacher B and Neveu A 1974 *Phys. Rev.* D **10** 4130
[2] Fayet P and Ferrara S 1977 *Phys. Rep.* **32** 249
[3] Murphy T and O'Raifeartaigh L 1983 *Nucl. Phys.* B **218** 484
[4] Zumino B 1975 *Nucl. Phys.* B **89** 535
[5] Imbimbo C and Mukhi S 1984 *Nucl. Phys.* B **247** 471
[6] Rajaraman R 1982 *Solitons and Instantons* (Amsterdam: North-Holland)
[7] Cahill K, Comtet A and Glauber R J 1976 *Phys. Lett.* **64B** 283
[8] Coleman S 1975 *Phys. Rev.* D **11** 2088
[9] Jackiw R and Woo G 1975 *Phys. Rev.* D **12** 1643
[10] Salam A and Strathdee J 1974 *Nucl. Phys.* B **76** 477
[11] Hull C M 1986 *Super Field Theory* (*Proc. Vancouver workshop*) ed G Kunstatter and H C Lee (New York: Plenum)
[12] Salomonson P 1982 *Nucl. Phys.* B **207** 350
[13] Witten E and Olive D 1978 *Phys. Lett.* **78B** 97
[14] Boya L J and Casahorrán J 1989 *Ann. Phys., NY* **196** 361
[15] Skyrme T H R 1962 *Nucl. Phys.* **31** 556
[16] Sakagami M 1982 *Nucl. Phys.* B **207** 430